MATH 239 Part I Notes

Angel Zhang

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1.1.5 Bijective Proofs

Surjective, Injective, Bijective

Let $f : A \to B$ be a function from a set A to a set B.

- The function f is **surjective** if for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$.
- The function f is **injective** if for every $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.
- The function f is **bijective** if it is both surjective and injective.

2.1 Binomial Theorem and Binomial Series

Theorem 2.2 (Binomial Theorem)

For any natural number $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Theorem 2.4 (Binomial Series/Negative Binomial Theorem)

For any positive integer $t \geq 1$,

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

2.2.1 Generating Series

Weight Function

Let A be a set. A function $w : A \to \mathbb{N}$ is a **weight function** if for every $n \in \mathbb{N}$ there are only finitely many elements $a \in A$ of weight n.

Generating Series

Let A be a set with a weight function $w : A \to \mathbb{N}$. The **generating series** of A with respect to w is

$$A(x) = \Phi_A^w(x) = \sum_{a \in A} x^{w(a)}$$

Proposition 2.7

Let A be a set with a weight function $w : A \to \mathbb{N}$, and let

$$\Phi_A(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

For every $n \in \mathbb{N}$, the number of elements of A of weight n is $a_n = |A_n|$.

3.2 Unambiguous Expressions

Unambiguous Expression

Let R be a regular expression that produces a rational language R. Then R is **unambiguous** if every string in R is produced exactly once by R. If an expression is not unambiguous, then it is **ambiguous**.

Theorem 3.13

Let R be a regular expression producing the rational language R and leading to the rational function R(x). If R is an unambiguous expression for R, then R(x) = $\Phi_R(x)$, the generating series for R with respect to length.

3.2.2 Block Decomposition

Proposition 3.17 (Block Decomposition)

The regular expressions $0^*(11^*00^*)1^*$ and $1^*(00^*11^*)0^*$ are unambiguous expressions for the set of all binary strings. They produce each binary string block by block.

4.3 Partial Fractions

Theorem 4.12 (Partial Fractions)

[See textbook]

4.4.1 The General Binomial Series

Negative Binomial Theorem

For a positive integer $t \geq 1$,

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Theorem 4.21 (The Binomial Series)

For any complex number $\alpha \in \mathbb{C}$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

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4.1 Definitions

Graph

A **graph** G is a finite nonempty set, $V(G)$, of objects, called **vertices**, together with a set $E(G)$, of unordered pairs of distinct vertices. The elements of $E(G)$ are called **edges**.

Adjacency

If $e = \{u, v\}$ then we say that u and v are **adjacent** vertices, and that edge e is **incident** with vertices u and v. We can also say that the edge e **joins** u and v. Vertices adjacent to a vertex u are called **neighbours** of u. The set of neighbours of u is denoted by $N(u)$.

Planar

A graph which can be represented with no edges crossing is said to be **planar**.

4.2 Isomorphism

Isomorphic

Two graphs G_1 and G_2 are **isomorphic** if there exists a bijection $f : V(G_1) \to V(G_2)$ such that vertices $f(u)$ and $f(v)$ are adjacent in G_2 if and only if u and v are adjacent in G_1. We might say that f preserves adjacency. The bijection f is called an **isomorphism**.

Isomorphism Class

The collection of graphs that are isomorphic to G forms the **isomorphic class** of G.

4.3 Degree

Degree

The number of edges incident with a vertex v is called the **degree** of v, and is denoted by $\deg(v)$.

Theorem 4.3.1 (Handshaking Lemma)

For any graph G,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Corollary 4.3.2

The number of vertices of odd degree in a graph is even.

Corollary 4.3.3

The average degree of a vertex in the graph G is

$$\frac{2|E(G)|}{|V(G)|}$$

Regular Graph

A graph in which every vertex has degree k, for some fixed k, is called a **k-regular graph** (or just a **regular graph**). The number of edges in a k-regular graph with n vertices is

$$\frac{nk}{2}$$

Complete Graph

A **complete graph** is one in which all pairs of distinct vertices are adjacent. The complete graph with p vertices is denoted by K_n, $n \geq 1$. The number of edges in K_n is

$$\binom{n}{2}$$

4.4 Bipartite Graphs

Complete Graph

A **complete graph** is one where every pair of vertices is an edge. A complete graph on n vertices is denoted K_n.

Bipartite Graph

A graph in which the vertices can be partitioned into two sets A and B, so that all edges join a vertex in A to a vertex in B, is called a **bipartite graph**, with **bipartition** (A, B).

Complete Bipartite Graph

The **complete bipartite graph** $K_{m,n}$ has all vertices in A adjacent to all vertices in B, with $|A| = m$, $|B| = n$. The number of edges in $K_{m,n}$ is mn

n-cube

For $n \geq 0$, the n**-cube** is the graph whose vertices are the binary strings of length n, and two strings are adjacent iff they differ in exactly one position.

Problem 4.2.2

The number of edges in a n-cube is $n2^{n-1}$.

Problem 4.2.3

The n-cube is bipartite.

4.6 Paths and Cycles

Subgraph

A **subgraph** of a graph G is a graph whose vertex set is a subset U of $V(G)$ and whose edge set is a subset of those edges of G that have both vertices in U.
If H is a subgraph of G and $V(H) = V(G)$, then we say H is a **spanning subgraph** of G.
If H is not equal to G, we say it is a **proper subgraph** of G.

Walk

A **walk** in a graph G from v_0 to v_n is an alternating sequence of vertices and edges of G which begins with v_0, ends with v_n and for $1 \leq i \leq n$, edge $e_i = \{v_{i-1}, v_i\}$. Such a walk can be called a v_0, v_n**-walk**.

Path

A **path** is a walk in which all the vertices are distinct. Note that all the edges are also distinct.

Theorem 4.6.2

If there is a walk from vertex x to vertex y in G, then there is a path from x to y in G.

Corollary 4.6.3

Let x, y, z be vertices of G. If there is a path from x to y in G and a path from y to z in G, then there is a path from x to z in G.

Cycle

A **cycle** is a connected graph that is regular of degree two. A cycle with n edges is called an n-**cycle** or a cycle of length n.
The shortest possible cycle in a graph is a 3-cycle, often called a **triangle**.

Theorem 4.6.4

If every vertex in G has degree at least 2, then G contains a cycle.

Girth

The **girth** of a graph G is the length of the shortest cycle in G, and is denoted by $g(G)$.

Hamilton Cycle

A spanning cycle is known as a **Hamilton cycle**.

4.8 Connectedness

Connected

A graph G is **connected** if for each two vertices x and y, there is a path from x to y.

Theorem 4.8.2.

Let G be a graph and let v be a vertex in G. If for each vertex w in G there is a path from v to w in G, then G is connected.

Problem 4.8.3

The n-cube is connected for all $n \geq 0$.

Component

A **component** of G is subgraph C of G such that

1. C is connected
2. No subgraph of G that properly contains C is connected

Theorem 4.8.5

A graph G is not connected if and only if there exists a proper nonempty subset X of $V(G)$ such that the cut induced by X is empty.

4.9 Eulerian Circuits

Eulerian Circuit

An Eulerian circuit of a graph G is a closed walk that contains every edge of G exactly once.

Theorem 4.9.2

Let G be a connected graph. Then G has an Eulerian circuit if and only if every vertex has even degree.

4.10 Bridges

If $e \in E(G)$, we denote by $G - e$ the graph whose vertex set is $V(G)$ and whose edge set is $E(G)\backslash\{e\}$, so $G - e$ is the graph obtained from G by deleting the edge e.

Edge

An edge e of G is a **bridge** if $G - e$ has more components than e. If G is connected, a bridge is an edge such that $G - e$ is not connected. Some text uses **cut-edge** as a synonym for bridge.

Lemma 4.10.2

If $e = xy$ is a bridge of a connected graph G, then $G - e$ has precisely two components. Furthermore, x and y are in different components.

Theorem 4.10.3

An edge e is a bridge of a graph G if and only if it is not contained in any cycle of G.

Corollary 4.10.4

If there are two distinct paths from vertex u to vertex v in G, then G contains a cycle. Equivalently, if a graph G has no cycles, then each pair of vertices is joined by at most one path.

5.1 Trees

Tree

A **tree** is a connected graph with no cycles.

Forest

A **forest** is a graph with no cycles.

Lemma 5.1.3

If u and v are vertices in a tree T, then there is a unique uv-path in T.

Lemma 5.1.4

Every edge of a tree T is a bridge.

Theorem 5.1.5

If T is a tree, then $|E(T)| = |V(T)| - 1$

Corollary 5.1.6

If G is a forest with k components, then $|E(G)| = |V(G)| - k$.

Leaf

A **leaf** in a tree is a vertex of degree 1.

Theorem 5.1.8

A tree with at least two vertices has at least two leaves.

Counting Leaves in a Tree

Let T be a tree and let n_k denote the number of vertices of degree k in T. Then the number of leaves in T, n_1, is

$$n_1 = 2 + \sum_{k \geq 3} (k-2) n_k$$

The formula above implies that if T contains a vertex of degree k, where $k \geq 3$, then

$$n_1 \geq 2 + (k-2) = k$$

5.2 Spanning Trees

A **spanning tree** is a spanning subgraph which is also a tree. Of all the spanning subgraphs, a spanning tree has the fewest edges while remaining connected.

Theorem 5.2.1

A graph G is connected if and only if it has a spanning tree.

Corollary 5.2.2

If G is connected, with p vertices and $p-1$ edges, then G is a tree.

Theorem 5.2.3

If T is a spanning tree of G and e is an edge not in T, then $T+e$ contains exactly one cycle C. Moreover, if e' is any edge on C, then $T+e-e'$ is also a spanning tree of G.

Theorem 5.2.4

If T is a spanning tree of G and e is an edge in T, then $T-e$ has 2 components. If e' is in the cut induced by one of the components, then $T-e+e'$ is also a spanning tree of G.

5.3 Characterizing Bipartite Graphs

Lemma 5.3.1

An odd cycle is not bipartite.

Theorem 5.3.2

A graph is bipartite if and only if it has no odd cycles.

5.6 Minimum Spanning Tree

In the minimum spanning tree (MST) problem, we are given a connected graph G and a weight function on the edges $w : E(G) \to \mathbb{R}$. The goal is to find a spanning tree in G whose total edge weight is minimized.

Theorem 5.6.1

Prim's algorithm produces a minimum spanning tree for G.

7.1 Planarity

Planar

A graph G is **planar** if it has a drawing in the plane so that its edges intersect only at their ends, and so that no two vertices coincide. The actual drawing is called a **planar embedding** of G, or a **planar map**. A graph is planar if and only if each of its components is planar.

Faces

A planar embedding partitions the plane into connected regions called **faces**. One of these regions, called the **outer face**, is unbounded.

Boundary

The subgraph formed by the vertices and edges in a face is called the **boundary** of the face.

Adjacent

Two faces are **adjacent** if they are incident with a common edge.

Boundary Walk

Assume that G is connected. As one moves around the entire perimeter of a face f, one encounters the vertices and edges in a fixed order, say

$$W_f = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n)$$

where $v_n = v_0$. This sequence is a closed walk of the graph G, and we call it the **boundary walk** of face f. (The boundary walk can start at any vertex, and can proceed around the perimeter in either a clockwise or counterclockwise direction.)

Degree

The number of edges in the boundary walk W_f is called the **degree** of the face f.

Note

A bridge of a planar embedding is incident with just one face, and is contained in the boundary walk of that face twice, once for each side. On the other hand, if e is an edge of a cycle of an embedding, e is incident with exactly two faces, and is contained in the boundary walk of each face precisely once. Every edge in a tree is a bridge, so a planar embedding of a tree T has a single face of degree $2|E(T)| = 2|V(T)| - 2$

Theorem 7.1.2 (Faceshaking Lemma)

If we have a planar embedding of a connected graph G with faces $f_1, \ldots, f_s$, then

$$\sum_{i=1}^{s} deg(f_i) = 2|E(G)|$$

Corollary 7.1.3

If the connected graph G has a planar embedding with f faces, then the average degree of a face in the embedding is

$$\frac{2|E(G)|}{f}$$

7.2 Euler's Formula

Every planar embedding of a given connected planar graph has the same number of faces, a fact that we can deduce from the following result.

Theorem 7.2.1 (Euler's Formula)

Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then

$$p - q + f = 2$$

7.4 Platonic Solids

The cube and the tetrahedron exhibit a great deal of symmetry. In particular, the faces have the same degree and the vertices have the same degree. Such polyhedra are called **platonic solids**. There are just five platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.

Platonic

A graph is **platonic** if it admits a planar embedding in which each vertex has the same degree $d \geq 3$ and each face has the same degree $d \geq 3$

Theorem 7.4.1

There are exactly five platonic graphs.

7.5 Nonplanar Graphs

Lemma 7.5.1

If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle.

Lemma 7.5.2

Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d, then

$$q \leq \frac{d(p-2)}{(d-2)}$$

Theorem 7.5.3

In a planar graph G with $p \geq 3$ vertices and q edges, we have

$$q \leq 3p - 6$$

Note on Theorem 7.5.3

If a graph does not satisfy the inequality, then it is not planar. However, the converse of the theorem is not necessarily true.

Corollary 7.5.4

K_5 is not planar.

Corollary 7.5.5

A planar graph has a vertex of degree at most five.

Theorem 7.5.6

In a bipartite planar graph G with $p \geq 3$ vertices and q edges, we have

$$q \leq 2p - 4$$

Lemma 7.5.7

$K_{3,3}$ is not planar.

7.6 Kuratowski's Theorem

We can use graphs based on K_5 and $K_{3,3}$ to certify that a graph is not planar.

Edge Subdivision

An **edge subdivision** of a graph G is obtained by applying the following operation, independently, to each edge of G: replace the edge by a path of length 1 or more. If the path has length $m > 1$, then there are $m-1$ new vertices and $m-1$ new edges created. If the path has length $m = 1$, then the edge is unchanged.

Note on Edge Subdivision

The operation of edge subdivision does not change planarity. If G is a planar graph, then all edge subdivisions of G are planar. If G is nonplanar, then all edge subdivions of G are nonplanar.

Theorem 7.6.1 (Kuratowski's Theorem)

A graph is not planar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$

7.7 Colouring and Planar Graphs

k-colouring

A k**-colouring** of a graph G is a function from $V(G)$ to a set of size k (whose elements are called **colours**), so that adjacent vertices always have different colours.

k-colourable

A graph with a k-colouring is called a k**-colourable** graph.

Theorem 7.7.2

A graph is 2-colourable if and only if it is bipartite.

Theorem 7.7.3

K_n is n-colourable, and not k-colourable for any $k < n$.

Theorem 7.7.4

Every planar graph is 6-colourable.

Edge Contraction

Let G be a graph and let $e = xy$ be an edge of G. The graph G/e obtained from G by **contracting** the edge e is the graph with vertex set $V(G) \setminus \{x, y\} \cup \{z\}$, where z is a new vertex, and edge set

$$\{u, v\} \in E(G) : \{u, v\} \cap \{x, y\} = \varnothing\} \cup \{\{u, z\} : u \notin \{x, y\}, \{u, w\} \in E(G) \text{ for some } w \in \{x, y\}\}$$

Intuitively, we can think of the operation of contracting e as allowing the length of e to decrease to 0, so that the vertices x and y are identified into a new vertex z. Any other vertex that was adjacent to one (or both) of x and y is adjacent to z in the new graph G/e.

Theorem 7.7.6

Every planar graph is 5-colourable.

Theorem 7.7.7 (Four Colour Theorem)

Every planar graph is 4-colourable.

7.8 Dual Planar Maps

Properties about Dual Planar Maps

Let G^* be the dual of a connected planar embedding G

1. A face of degree k in G becomes a vertex of degree k in G^*
2. A vertex of degree j in G becomes a face of degree j in G^*
3. $(G^*)^*$ and G are the same graph.
4. G^* may be a multigraph rather than a graph.
5. The Four Colour Theorem for colouring vertices in planar graphs is equivalent to the Four Colour Theorem for colouring faces in planar embeddings, via duality.

8.1 Matching

Matching

A **matching** in a graph G is a set M of edges of G such that no two edges in M have a common end.

Saturated

We say that a vertex v of G is **saturated** by M, or that M saturates v, if v is incident with an edge in M.

Maximum Matching

The largest matching in a graph G is called a **maximum matching** of G.

Perfect Matching

A matching that saturates every vertex is called a **perfect matching**.

Augmenting Path

An **augmenting path** with respect to M is an alternating path joining two distinct vertices, neither of which is saturated by M. Note that augmenting paths have odd length because they begin and end with non-matching edges.

Lemma 8.1.1

If M has an augmenting path, it is not a maximum matching.

8.2 Covers

Cover

A **cover** of a graph G is a set C of vertices such that ever edge of G has at least one end in C.

Lemma 8.2.1

If M is a matching of G and C is a cover of G, then $|M| \leq |C|$.

Lemma 8.2.2

If M is a matching of G and C is a cover of G and $|M| = |C|$, then M is a maximum matching and C is a minimum cover.

8.3 König's Theorem

Theorem 8.3.1 (König's Theorem)

In a bipartite graph, the maximum size of a matching is the minimum size of a cover. That is, if M is a maximum matching and C is a minimum cover, then $|M| = |C|$.

Bipartite Matching Algorithm (XY-Construction)

Given a bipartite graph G with bipartition (A, B), and a matching M of G.

1. Let X_0 be the est of all unsaturated vertices in A. Set $X = X_0$ and $Y = \varnothing$.
2. For all neighbours of X in B currently not in Y.
   (a) If one such vertex is unsaturated, then we have found an augmenting path. Obtain a larger matching by swapping edges in the augmenting path. Go to step 1.
   (b) If all such vertices are saturated, then put all of them in Y. Add their matching neighbours to X. Go to step 2.
   (c) If no such vertices exist, then stop. Our matching is maximum with minimum cover $Y \cup (A \backslash X)$.

8.4 Applications of König's Theorem

Neighbour Set

For any subset D of vertices of a graph G, the **neighbour set**, $N(D)$, is defined as the set of all vertices $v \in V(G)$ such that there exists $u \in D$ with $\{u, v\} \in E(G)$. ·

Theorem 8.4.1 (Hall's Theorem)

A bipartite graph G with bipartition A, B has a matching saturating every vertex in A if and only if every subset D of A satisfies

$$|N(D)| \geq |D|$$

8.6 Perfect Matching in Bipartite Graph

We can use Hall's Theorem to obtain a condition for a bipartite graph to have a perfect matching.

Corollary 8.6.1

A bipartite graph G with bipartition A, B has a perfect matching if and only if $|A| = |B|$ and every subset D of A satisfies

$$|N(D)| \geq |D|$$

Theorem 8.6.2

If G is a k-regular bipartite graph with $k \geq 1$, then G has a perfect matching.

Note

Theorem 8.6.2 works even if G contains multiple edges.